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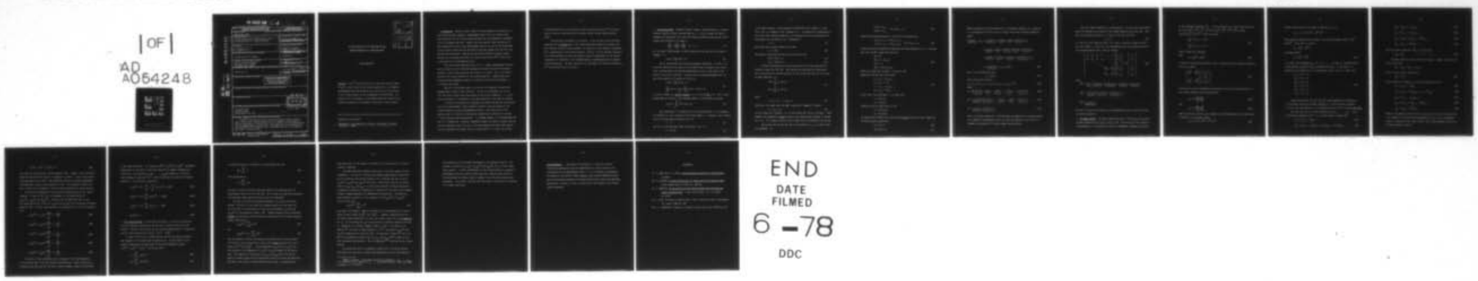

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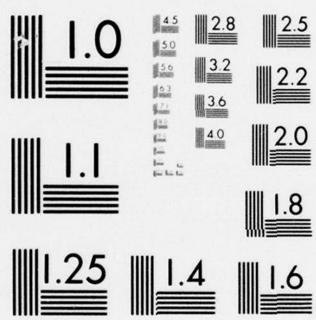

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ON THE MULTIPLICITY OF SOLUTIONS OF THE INVERSE PROBLEM FOR A VIBRATING BEAM*

VICTOR BARCILON†

Abstract: The 2^{N-1} fold multiplicity of solutions found by Boley & Golub in their study of the inverse problem for N x N symmetric, pentadiagonal matrices contrasts with the unicity of the solution of the inverse problem for an inhomogeneous, discrete beam. The reason for this discrepancy is elucidated and can be traced to the different properties of the spectral data used in the two cases.

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†Department of the Geophysical Sciences, University of Chicago, Chicago, IL. 60637.

1. Introduction. Recently, Boley & Golub [1] have examined the problem of reconstructing an NxN, symmetric, pentadiagonal matrix from its spectral data. The formulation of this inverse problem follows the line pioneered by Hochstadt [2] for tridiagonal matrices. Namely, the given spectral data are made up by the eigenvalues of the original pentadiagonal matrix, as well as the eigenvalues of two closely related matrices obtained by deleting respectively the first one and two rows and columns. Whereas the equivalent procedure yielded a unique solution for tridiagonal matrices [2], Boley & Golub found a 2^{N-1} multiplicity of solutions for the pentadiagonal case.

This result might seem puzzling at first. Indeed, pentadiagonal matrices can be viewed as finite difference analogues of certain 4th order differential operators, such as that governing the vibration of a beam. And, in my investigation of the inverse problem for the discrete beam [3], I showed that given three so-called sympathetic spectra, the solution of the inverse problem, if it existed, was unique.

The aim of the present paper is to clarify this apparent contradiction between Boley & Golub's result and mine. The key to the paradox lies in the fact that Boley & Golub used three spectra which are not sympathetic, i.e. which are not equivalent in information to the data contained in the impulse response.

In order to illustrate this important point about the need for the spectral trio to be interrelated, I shall consider a specific inverse problem for the vibrating beam with three non-sympathetic spectra. More specifically, the spectral data will consist of the natural frequencies of vibration of the beam in the following three configurations: (i) clamped-clamped, (ii) clamped-supported and (iii) clamped-free. Eminently reasonable though this choice of spectra may be from the engineering point of view, it has the drawback that these three spectra are not sympathetic and cannot insure a unique solution. In fact, for the Nth discretized version of this problem, we shall see that there are 2^{N-1} solutions, thus arriving at a multiplicity of solutions similar to that found by Boley & Golub.

The outline of the paper is as follows. In §2, we shall derive the basic equations for a discrete beam: this simple mechanical system will provide valuable insights into the inverse problem. §3 contains all the necessary ingredients for a consideration of the inverse problem. Finally, in §4 we consider the particular inverse problem for which the spectral data are associated with the natural frequencies of vibrations in the clamped/clamped, clamped/supported and clamped/free configurations. We shall show that with such data, the inverse problem has a 2^{N-1} fold multiplicity of solutions.

2. <u>The discrete beam.</u> Consider a beam of length L characterized by a variable flexural rigidity EI and a variable density ρ. If $y(x,t)$ stands for the displacement of the central line, then the classical equation governing the infinitesimal oscillations of this beam is

$$-\frac{\partial^2}{\partial x^2} EI \frac{\partial^2 y}{\partial x^2} = \rho \frac{\partial^2 y}{\partial t^2} \quad , \; 0 < x < L. \tag{1}$$

For the sake of definiteness, we shall assume that the left end of the beam is clamped, i.e.

$$y(0,t) = \frac{\partial}{\partial x} y(0,t) = 0 \tag{2}$$

Let us now consider the following conceptual experiment: at time $t = 0$, we strike impulsively the right end of the beam which is assumed to be stress-free. We then monitor the ensuing displacement as well as the slope of the central line of this right end. To carry out this program mathematically, we must solve equation (1) subject to (2) and to

$$\begin{gathered} y(x,0) = \frac{\partial}{\partial t} y(x,0) = 0, \\ \frac{\partial^2}{\partial x^2} y(L,t) = \frac{\partial}{\partial x} EI \frac{\partial^2}{\partial x^2} y(L,t) - F\delta(t) = 0, \end{gathered} \tag{3}$$

in order to deduce the <u>impulse response</u>, viz. $y(L,t)$ and $\frac{\partial}{\partial x} y(L,t)$. This is best accomplished by working in the frequency domain; to that effect, we define

$$u(x,\omega) = \int_0^\infty e^{i\omega t}\, y(x,t)\, dt. \tag{4}$$

More importantly, in preparation for the discretization, it is convenient to transform (1) into a system of first order equations. Therefore, let us define $\theta(x,\omega)$ to be the slope of central line, i.e.

$$u' = \theta \tag{5}$$

and $\tau(x,\omega)$ to be the moment about the central line, i.e.

$$\theta' = f(x)\tau. \tag{6}$$

In the above formulas, a prime denotes differentiation with respect to x and $f(x) = 1/EI$ is a measure of the "limpness" at x. To complete our transformation to a first order system of equations, we introduce $\phi(x,\omega)$ which represents the force applied to the central line. Consequently,

$$\tau' = \phi \tag{7}$$

With these new variables, Newton's law reads

$$\phi' = -\omega^2 \rho u. \tag{8}$$

The boundary conditions (2)-(3) can now be written thus:

$$u(0) = \theta(0) = 0, \tag{9}$$

$$\tau(L) = \phi(L) - F = 0. \tag{10}$$

We shall be concerned with the discrete version of the system (5)-(8) subject to conditions (9)-(10). The simplest and most physically meaningful way to discretize this problem consists in restricting $\rho(x)$ and $f(x)$ to be sums of delta functions, viz.

$$\begin{aligned} \rho(x) &= \sum_{i=1}^{N} m_i \; \delta(x-x_i) \quad , \\ f(x) &= \sum_{i=1}^{N} f_i \; \delta(x-x_i) \quad . \end{aligned} \tag{11}$$

where

$$0 = x_o < x_1 < \ldots < x_N = L. \tag{12}$$

Physically, this means that the beam is made up of segments of lengths

$$\ell_i = x_{i+1} - x_i \tag{13}$$

of zero weight and "limpness", i.e. of zero weight and infinite stiffness. These segments are connected by joints of mass m_i and concentrated limpness f_i located at $x = x_i$. For instance, we can think of these joins as clothespin-like devices.

We can see from (6) and (8) that in the interval (x_{i-1}, x_i), both θ and ϕ are constants, i.e.

$$\left.\begin{array}{l}\theta(x,\omega) = \theta_{i-1}, \\ \phi(x,\omega) = \phi_{i-1}\end{array}\right\} \quad \text{for } x\varepsilon(x_{i-1}, x_i) \tag{14}$$

Substituting these expressions in (5) and (7) we deduce that

$$\left.\begin{array}{l}u(x,\omega) = u_{i-1} + (x-x_{i-1})\ \theta_{i-1} \\ \tau(x,\omega) = \tau_{i-1} - (x-x_{i-1})\ \phi_{i-1}\end{array}\right\} \quad \text{for } x\varepsilon(x_{i-1}, x_i) \tag{15}$$

We get the discrete equations by examining the jump conditions at x_i. As can be seen from (5)-(8), these conditions are:

$$\begin{aligned}
[u]_i &= 0 \\
[\theta]_i &= f_i\ \tau(x_i,\omega) \\
[\tau]_i &= 0 \\
[\phi]_i &= -\omega^2 m_i\ u(x_i,\omega)
\end{aligned} \tag{16}$$

where $[u]_i$ stands for $u(x_i+0,\omega) - u(x_i-0,\omega)$, etc.

Replacing (14)-(15) into (16) yields

$$\begin{aligned}
u_i &= u_{i-1} + \ell_{i-1}\theta_{i-1} \quad , \\
\theta_i &= \theta_{i-1} + f_i\tau_i \quad , \\
\tau_i &= \tau_{i-1} - \ell_{i-1}\phi_{i-1} \quad , \\
\phi_i &= \phi_{i-1} - \omega^2 m_i u_i \quad .
\end{aligned} \tag{17}$$

Since u and τ are continuous, it is clear that

$$u_i = u(x_i,\omega) \ ,$$

$$\tau_i = \tau(x_i,\omega),$$

whereas by the very definition in (14)

$$\theta_i = \theta(x_i+0,\omega),$$

$$\phi_i = \phi(x_i+0,\omega).$$

To complete the formulation of the discrete <u>direct</u> problem, we shall impose the following boundary conditions:

$$u_o = \theta_o = 0, \tag{18}$$

$$t_N = \phi_N - F = 0. \tag{19}$$

Before proceeding with the computation of the impulse response, viz. u_N and θ_N, it is perhaps worth writing (17) as a single forth order difference equation, namely

$$\frac{1}{f_{i+1}\ell_i\ell_{i+1}}\,u_{i+2} - \left\{\frac{1}{f_{i+1}\ell_i\ell_{i+1}} + \frac{1}{f_{i+1}\ell_i^{\,2}} + \frac{1}{f_i\ell_i^{\,2}} + \frac{1}{f_i\ell_{i-1}\ell_i}\right\} u_{i+1}$$

$$+ \left\{\frac{1}{f_{i+1}\ell_i^{\,2}} + \frac{1}{f_i\ell_i^{\,2}} + \frac{2}{f_i\ell_{i-1}\ell_i} + \frac{1}{f_i\ell_{i-1}^{\,2}} + \frac{1}{f_{i-1}\ell_{i-1}^{\,2}}\right\} u_i$$

$$- \left\{\frac{1}{f_i\ell_{i-1}\ell_i} + \frac{1}{f_i\ell_{i-1}^{\,2}} + \frac{1}{f_{i-1}\ell_{i-1}^{\,2}} + \frac{1}{f_{i-1}\ell_{i-2}\ell_{i-1}}\right\} u_{i-1}$$

$$+ \frac{1}{f_{i-1}\ell_{i-2}\ell_{i-1}}\,u_{i-2} = \omega^2 m_i u_i. \tag{18}$$

Now, if we introduce the field

$$v_i = m_i^{\frac{1}{2}} u_i \tag{19}$$

then (18) can be written as

$$c_{i-2}\,v_{i-2} + b_{i-1}v_{i-1} + a_i v_i + b_i v_{i+1} + c_i v_{i+2} = \omega^2 v_i \tag{20}$$

where

$$a_i = \frac{1}{m_i}\left[\frac{1}{f_{i+1}\ell_i^{\,2}} + \frac{1}{f_i\ell_i^{\,2}} + \frac{2}{f_i\ell_{i-1}\ell_i} + \frac{1}{f_i\ell_{i-1}^{\,2}} + \frac{1}{f_{i-1}\ell_{i-1}^{\,2}}\right], \tag{21a}$$

$$b_i = \frac{-1}{(m_i m_{i+1})^{\frac{1}{2}}}\left[\frac{1}{f_{i+1}\ell_i\ell_{i+1}} + \frac{1}{f_{i+1}\ell_i^{\,2}} + \frac{1}{f_i\ell_i^{\,2}} + \frac{1}{f_i\ell_{i-1}\ell_i}\right], \tag{21b}$$

$$c_i = \frac{1}{(m_i m_{i+2})^{\frac{1}{2}}}\left[\frac{1}{f_{i+1}\ell_i\ell_{i+1}}\right]. \tag{21c}$$

Thus, the inverse problem for a discrete beam corresponds to an inverse problem for a symmetric pentadiagonal matrix in which the elements of the first off-diagonals are negative, all other elements being positive.

One final remark regarding the discretization. No particular significance should be attached to the choice of the clamped boundary conditions (18). However, having made that choice, it is easy to see from (17) that

$$u_1 = 0, \tag{22}$$

i.e. the first joint does not move! As a result, the mass m_1 does not enter into the problem. In fact, for this clamped/free case, the eigenvalue problem for the pentadiagonal matrix looks as follows:

$$\begin{bmatrix} a_2 & b_2 & c_2 & 0 & \cdots & \\ b_2 & a_3 & b_3 & c_3 & \cdots & \\ & \ddots & & & & \\ & & & \ddots & a^f_{N-1} & b^f_{N-1} \\ & & & & b^f_{N-1} & a^f_N \end{bmatrix} \begin{bmatrix} v_2 \\ v_3 \\ \vdots \\ v_{N-1} \\ v_N \end{bmatrix} = \omega^2 \begin{bmatrix} v_2 \\ v_3 \\ \vdots \\ v_{N-1} \\ v_N \end{bmatrix} \tag{23}$$

where

$$a^f_{N-1} = \frac{1}{m_{N-1}} \left[\frac{1}{f_{N-1}\ell_{N-1}{}^2} + \frac{2}{f_{N-1}\ell_{N-2}\ell_{N-1}} + \frac{1}{f_{N-1}\ell_{N-2}{}^2} + \frac{1}{f_{N-2}\ell_{N-2}{}^2} \right]$$

$$b^f_{N-1} = \frac{-1}{(m_{N-1}m_N)^{\frac{1}{2}}} \left[\frac{1}{f_{N-1}\ell_{N-1}{}^2} + \frac{1}{f_{N-1}\ell_{N-2}\ell_{N-1}} \right],$$

$$a^f_N = \frac{1}{m_N f_{N-1}\ell_{N-1}{}^2} .$$

We shall not pursue the problem via the pentadiagonal matrix since this tends to shroud the physics.

3. The wedge product. The direct problem consists in finding u_N and θ_N given the beam characteristics $\{\ell_i\}_0^N$, $\{m_i\}_2^N$, $\{f_i\}_1^N$. The solution to this problem is straightforward. We introduce two linearly independent fundamental solutions

of the difference equations (17). Using a superscript to label these solutions, we define $[u_i^{(1)}, \theta_i^{(1)}, \tau_i^{(1)}, \phi_i^{(1)}]$ and $[u_i^{(2)}, \theta_i^{(2)}, \tau_i^{(2)}, \phi_i^{(2)}]$ or more succintly $\underset{\sim}{u}_i^{(1)}$ and $\underset{\sim}{u}^{(2)}$ in such a way that

$$\underset{\sim}{u}_0^{(1)} = [0 \quad 0 \quad 1 \quad 0]^T \ ,$$
$$\underset{\sim}{u}_0^{(2)} = [0 \quad 0 \quad 0 \quad 1]^T \ , \tag{24}$$

where T stands for transpose.

Then

$$\underset{\sim}{u}_i = a\underset{\sim}{u}_i^{(1)} + b\underset{\sim}{u}_i^{(2)} . \tag{25}$$

In order to evaluate the constants a and b, we make use of the boundary conditions (19) at x_N, namely

$$\begin{bmatrix} \tau_N^{(1)} & \tau_N^{(2)} \\ \\ \phi_N^{(1)} & \phi_N^{(2)} \end{bmatrix} \begin{bmatrix} a \\ \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ \\ F \end{bmatrix} . \tag{26}$$

Omitting the trivial intermediary calculations we find that the constituents of the impulse response can be expressed thus:

$$u_N = -F \frac{\{u_N, \tau_N\}}{\{\tau_N, \phi_N\}} \ , \tag{27a}$$

and

$$\theta_N = -F \frac{\{\theta_N, \tau_N\}}{\{\tau_N, \phi_N\}} \ , \tag{27b}$$

where from now on we shall use a bracket as a shorthand notation for determinants of certain 2 x 2 matrices, namely

$$\{a, b\} = a^{(1)} \ b^{(2)} - a^{(2)} \ b^{(1)} \tag{28}$$

Bracket expressions will be formed for each point x_i, e.g.

$$\{u_i, \tau_i\} = u_i^{(1)} \tau_i^{(2)} - u_i^{(2)} \tau_i^{(1)} \quad .$$

These bracket expressions are related to the so-called wedge product of $\underset{\sim}{u}_i^{(1)}$ and $\underset{\sim}{u}_i^{(2)}$. Given the 4 x 2 matrix

$$\underset{\sim}{U}_i = [\underset{\sim}{u}_i^{(1)} \quad \underset{\sim}{u}_i^{(2)}] \quad ,$$

the wedge product

$$\underset{\sim}{Z}_i = \underset{\sim}{u}_i^{(1)} \wedge \underset{\sim}{u}_i^{(2)} \tag{29}$$

is a vector with components $Z_{\alpha,i}$ ($\alpha = 1, 2, \ldots, 6$) equal to the determinants of the matrices obtained from $\underset{\sim}{U}_i$ by deleting two rows. If the label α is attached to the chosen rows in a lexicographic manner, then it is clear that

$$\begin{aligned} Z_{1,i} &= \{u_i, \theta_i\} \\ Z_{2,i} &= \{u_i, \tau_i\} \\ Z_{3,i} &= \{u_i, \phi_i\} \\ Z_{4,i} &= \{\theta_i, \tau_i\} \\ Z_{5,i} &= \{\theta_i, \phi_i\} \\ Z_{6,i} &= \{\tau_i, \phi_i\} \end{aligned} \tag{30}$$

Rather than working with $\underset{\sim}{u}_i^{(1)}$ and $\underset{\sim}{u}_i^{(2)}$ whose components are polynomials in ω^2 and then forming the bracket expressions, a risky numerical procedure in view of the cancellations, it is preferable to work with the brackets _ab initio_.

Our next task then is to derive equations for $Z_{1,i}, \ldots, Z_{6,i}$. Using the difference equations (17), it is a simple matter to deduce that

$$Z_{1,i} = Z_{1,i-1} + f_i Z_{2,i} \quad , \tag{31a}$$

$$Z_{2,i} = Z_{2,i-1} - \ell_{i-1} Z_{3,i-1} + \ell_{i-1} Z_{4,i-1} - \ell_{i-1}^2 Z_{5,i-1} \quad , \tag{31b}$$

$$Z_{3,i} = Z_{3,i-1} + \ell_{i-1} Z_{5,i-1} \quad , \tag{31c}$$

$$Z_{4,i} = Z_{4,i-1} - \ell_{i-1} Z_{5,i-1} \quad , \tag{31d}$$

$$Z_{5,i} = Z_{5,i-1} + f_i Z_{6,i-1} + \omega^2 m_i Z_{1,i} \quad , \tag{31e}$$

$$Z_{6,i} = Z_{6,i-1} + \omega^2 m_i Z_{2,i} \quad . \tag{31f}$$

From the boundary conditions (24), it follows that

$$Z_{\alpha,o} = \delta_{\alpha 6} \qquad \alpha = 1, 2, \ldots, 6 \quad . \tag{32}$$

The above equations can be simplified slightly. Indeed, from (31c) and (31d), we see that

$$Z_{3,i} + Z_{4,i} = Z_{3,i-1} + Z_{4,i-1}$$

which, in view of (32), implies that

$$Z_{3,i} = -Z_{4,i} \quad . \tag{33}$$

We can therefore eliminate $Z_{3,i}$ and rewrite equations (31) as

$$Z_{1,i} = Z_{1,i-1} + f_i Z_{2,i} \quad , \tag{34a}$$

$$Z_{2,i} = Z_{2,i-1} + 2\ell_{i-1} Z_{4,i-1} - \ell_{i-1}^{\;2} Z_{5,i-1} \quad , \tag{34b}$$

$$Z_{4,i} = Z_{4,i-1} - \ell_{i-1} Z_{5,i-1} \, , \tag{34c}$$

$$Z_{5,i} = Z_{5,i-1} + f_i Z_{6,i-1} + \omega^2 m_i Z_{1,i} \quad , \tag{34d}$$

$$Z_{6,i} = Z_{6,i-1} + \omega^2 m_i Z_{2,i} \quad . \tag{34e}$$

These are the equations which are best suited for solving the inverse problem. It is important to realize that the components of $Z_{\alpha,i}$ are not independent. Indeed, in addition to the linear identity (33), these components satisfy the following quadratic identity:

$$Z_{1,i}Z_{6,i} - Z_{4,i}^2 - Z_{2,i}Z_{5,i} = 0 \tag{35}$$

This identity can be derived from the equations (34). However, such a derivation hides the fact that this identity is algebraic in nature. It has a long history which goes back to work done in the eighteenth century on "vanishing aggregates of determinants" (see e.g. [4], bottom of p. 50). This quadratic identity will play a crucial role in whether or not the inverse problem has a unique solution.

We close this paragraph by returning to the determination of the impulse response. In view of (27), this is tantamount to the determination of $Z_{2,N}(\omega^2)$, $Z_{4,N}(\omega^2)$ and $Z_{6,N}(\omega^2)$. Starting with the conditions (32), we can find these quantities, as well as $Z_{1,N}(\omega^2)$ and $Z_{5,N}(\omega^2)$, by solving the difference equations (34). For $N \geq 2$, these quantities are polynomials in ω^2 of the following form

$$Z_{1,N}(\omega^2) = Z_{1,N}(0) \prod_{n=1}^{N-2} \left(1 - \frac{\omega^2}{\kappa_n^2}\right) \quad , \tag{36a}$$

$$Z_{2,N}(\omega^2) = Z_{2,N}(0) \prod_{n=1}^{N-2} \left(1 - \frac{\omega^2}{\lambda_n^2}\right) \quad , \tag{36b}$$

$$Z_{4,N}(\omega^2) = Z_{4,N}(0) \prod_{n=1}^{N-2} \left(1 - \frac{\omega^2}{\mu_n^2}\right) \quad , \tag{36c}$$

$$Z_{5,N}(\omega^2) = Z_{5,N}(0) \prod_{n=1}^{N-1} \left(1 - \frac{\omega^2}{\nu_n^2}\right) \quad , \tag{36d}$$

$$Z_{6,N}(\omega^2) = Z_{6,N}(0) \prod_{n=1}^{N-1} \left(1 - \frac{\omega^2}{\xi_n^2}\right) \quad . \tag{36e}$$

The zeros of these polynomials are the squares of the eigenfrequencies of the discrete beam in various vibrating configurations. These vibrating configurations are such that the left end is always clamped, whereas the conditions at the right end differ. For instance $\{\kappa_n\}_1^{N-2}$, $\{\lambda_n\}_1^{N-2}$ and $\{\xi_n\}_1^{N-1}$ correspond respectively to the cases in which the right end is clamped, supported and stress free. The constants $Z_{1,N}(0)$, . . ., $Z_{6,N}(0)$ appearing in (36) are easily deduced by setting $\omega^2 = 0$ in (34) and solving the resulting difference equations. This calculation shows that

$$Z_{1,N}(0) = Q \equiv -\sum_{i=2}^{N} f_i \sum_{j=1}^{i-1} (x_i - x_j)^2 f_j, \quad (N \geq 2) \tag{37a}$$

$$Z_{2,N}(0) = -P_2 \equiv -\sum_{i=1}^{N-1} (x_N - x_i)^2 f_i, \quad (N \geq 2) \tag{37b}$$

$$Z_{4,N}(0) = -P_1 \equiv -\sum_{i=1}^{N-1} (x_N - x_i) f_i, \quad (N \geq 2) \tag{37c}$$

$$Z_{6,N}(0) = 1 . \tag{37e}$$

4. <u>The inverse problem</u>. As previously mentioned, in view of our particular choice of boundary conditions at the left end, m_1 does not enter into the problem. Therefore, the solution of the inverse problem consists in recovering the 3N-1 beam characteristics $\{m_i\}_2^N$, $\{f_i\}_1^N$, $\{\ell_i\}_0^{N-1}$.

In order to arrive at a unique solution, we must use impulse response data together with two additional constraints [3]. We have seen that the impulse response data are equivalent to the three sympathetic spectra $\{\lambda_n\}_1^{N-2}$, $\{\mu_n\}_1^{N-2}$, $\{\xi_n\}_1^{N-1}$ and the two sums

$$P_1 = \sum_{i=1}^{N-1} (x_N - x_i) f_i \quad , \tag{38}$$

$$P_2 = \sum_{i=1}^{N-1} (x_N - x_i)^2 f_i \quad . \tag{39}$$

To these 3N-3 pieces of information, we add another sum, say

$$P_0 = \sum_{i=1}^{N-1} f_i \tag{40}$$

and a normalization

$$x_L \equiv \sum_{i=0}^{N-1} \ell_i = L \ . \tag{41}$$

Of course, we could translate these requirements into analogous ones for pentadiagonal matrices of the type (23). But, without the underlying connection with the beam, these conditions would not be very meaningful.

We shall not review the procedure necessary for solving the inverse problem. Suffice it to say that this procedure consists of two steps [3]. The first step is the evaluation of the missing polynomials $Z_{1,N}(\omega^2)$ and $Z_{5,N}(\omega^2)$ via the quadratic identity (35). These polynomials can be determined uniquely provided that the given data are associated with the impulse response. Indeed, substituting

$$Z_{1,N}(\omega^2) = \sum_{\kappa=0}^{N-2} q_\kappa \, \omega^{2\kappa} \tag{42}$$

and

$$Z_{5,N}(\omega^2) = P_0 + \sum_{\kappa=1}^{N-1} p_\kappa \omega^{2\kappa} \tag{43}$$

into the quadratic identity and equating the coefficients of various powers of ω^2 equal to zero we can derive a set of 2N-2 linear equations for the unknowns $\{p_\kappa\}_1^{N-1}$ and $\{q_\kappa\}_0^{N-2}$. Having determined $Z_{1,N}(\omega^2)$ and $Z_{5,N}(\omega^2)$ and thus knowing all the components of $Z_{\alpha,N}(\omega^2)$ we can proceed with the second step. This consists in finding m_N, f_N, ℓ_{N-1} and $Z_{\alpha,N-1}(\omega^2)$ from (34) by means of suitable operations with polynomials such as divisions and additions. Once again, the results of these operations are unique. By repeating the same operations for the successive intervals the unique solution to inverse problem is obtained.

The above mentioned procedure breaks down if the given spectra are not sympathetic. In order to illustrate the changes brought about by data which are not related to the impulse response, let us consider the case in which $Z_{1,N}(\omega^2)$, $Z_{2,N}(\omega^2)$ and $Z_{6,N}(\omega^2)$ are given. Referring to (30) for the definitions of $Z_{1,N}$, $Z_{2,N}$ and $Z_{6,N}$ it is clear that the zeros of these polynomials correspond to the natural frequencies of vibrations of the beam in the clamped/clamped, clamped/supported and clamped/free configurations. Following the same procedure as before, we first attempt to find $Z_{5,N}(\omega^2)$ and $Z_{4,N}(\omega^2)$ which we write thus:

$$Z_{4,N}(\omega^2) = \sum_{\kappa=0}^{N-2} s_\kappa \, \omega^{2\kappa} \tag{44}$$

Once again, the quadratic identity provides us with 2N-2 equations for the unknowns at hand, namely $\{q_\kappa\}_0^{N-2}$ and $\{s_\kappa\}_0^{N-2}$. However, these equations are no longer linear equations for q_κ and s_κ but rather linear for q_κ and quadratic for s_κ! By eliminating the q_κ's we can deduce N-1 quadratic equations for the s_κ. Appealing to a theorem of Bézout† (see e.g. [5], p. 10) there are in general 2^{N-1} solutions to these equations i.e. 2^{N-1} polynomials $Z_{4,N}(\omega^2)$ and $Z_{5,N}(\omega^2)$ compatible with the given data. For each one of these 2^{N-1} pairs, we can find corresponding values of m_N, f_N, ℓ_{N-1} and $Z_{\alpha,N-1}(\omega^2)$ by means of the usual polynomial manipulations. Thus, we generate 2^{N-1} solutions to the inverse problem.

The second definition of sympathetic spectra [3] is now quite natural. Such spectra are the zeros of those three polynomials for which the remaining

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†Bézout's theorem: N polynomial equations of degrees $n_1, n_2, \ldots, n_N$ in N variables have in general $n_1 n_2 \ldots n_N$ common solutions. When the number is greater, it is infinite.

two polynomials can be uniquely determined by the quadratic identity. For instance, the zeros of $Z_{1,N}(\omega^2)$, $Z_{4,N}(\omega^2)$ and $Z_{2,N}(\omega^2)$ form a trio of sympathetic spectra. In their investigation of the inverse problem for symmetric, pentadiagonal matrices, Boley & Golub chose their spectral data primarily for convenience and without regard to whether or not the three spectra were sympathetic. As a result, the fact that they found a multiplicity of solutions is no longer surprising.

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